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Distribution of Particles Which Produces a "Smart" Material

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If $A_q(\beta, \alpha, k)$ is the scattering amplitude, corresponding to a potential $q \in L^2(D)$, where $D \subset \mathbb{R}^3$ is a bounded domain, and $e^{ik\alpha\cdot x}$ is the incident plane wave, then we call the radiation pattern the function $A(\beta) := A_q(\beta, \alpha, k)$, where the unit vector α, the incident direction, is fixed, β is the unit vector in the direction of the scattered wave, and $k > 0$, the wavenumber, is fixed. It is shown that any function $f(\beta) \in L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$, can be approximated with any desired accuracy by a radiation pattern: $\|f(\beta) - A(\beta)\|_{L^2(S^2)} < \epsilon$, where $\epsilon > 0$ is an arbitrary small fixed number. The potential q, corresponding to $A(\beta)$, depends on f and ϵ, and can be calculated analytically. There is a one-to-one correspondence between the above potential and the density of the number of small acoustically soft particles $D_m \subset D$, $1 \le m \le M$, distributed in an a priori given bounded domain $D \subset \mathbb{R}^3$. The geometrical shape of a small particle D_m is arbitrary, the boundary S_m of D_m is Lipschitz uniformly with respect to m. The wave number k and the direction α of the incident upon D plane wave are fixed. It is shown that a suitable distribution of the above particles in D can produce the scattering amplitude $A(\alpha', \alpha)$, $\alpha', \alpha \in S^2$, at a fixed $k > 0$, arbitrarily close in the norm of $L^2(S^2 \times S^2)$ to an arbitrary given scattering amplitude $f(\alpha', \alpha)$, corresponding to a real-valued potential $q \in L^2(D)$, i.e., corresponding to an arbitrary refraction coefficient in D.

KEY WORDS: scattering by small bodies, scattering amplitude, radiation pattern, nanotechnology, inverse scattering.

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1. INTRODUCTION

Let $D \subset \mathbb{R}^3$ be a bounded connected domain with Lipschitz boundary S.

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The scattering of an acoustic plane wave $u_0 = u_0(x) = e^{ik\alpha\cdot x}$, incident upon D, is described by the problem:

$$[\nabla^2 + k^2 n_0(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{1}$$

$$u = u_0(x) + v, \tag{2}$$

$$v = A(\alpha', \alpha)\,\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \qquad r := |x| \to \infty, \qquad \frac{x}{r} := \alpha'. \tag{3}$$

The coefficient $A(\alpha', \alpha)$ is called the scattering amplitude, $k > 0$ is the wave number, which is assumed fixed throughout the paper, and the dependence of A on k is not shown by this reason, $\alpha \in S^2$ is the direction of the incident plane wave, α' is the direction of the scattered wave, $n_0(x)$ is the known refraction coefficient in D, $n_0(x) = 1$ in $D' := \mathbb{R}^3 \setminus D$, and v is the scattered field.

Let D_m, $1 \le m \le M$, be a small particle, i.e.,

$$k_0 a \ll 1, \qquad \text{where} \qquad a = \frac{1}{2}\max_{1\le m\le M} \operatorname{diam} D_m, \qquad k_0 = k \max_{x\in D} |n_0(x)|. \tag{4}$$

The geometrical shape of D_m is arbitrary. We assume that D_m is a Lipschitz domain uniformly with respect to m. This is a technical assumption which can be relaxed. It allows one to use the properties of the electrostatic potentials. Denote

$$d := \min_{m\ne j} \operatorname{dist}(D_m, D_j). \tag{5}$$

Assume that

$$a \ll d. \tag{6}$$

We do not assume that $d \gg \lambda_0$, that is, that the distance between the particles is much larger than the wavelength. Under our assumptions, it is possible that there are many small particles on the distances of the order of the wavelength.

The particles are assumed acoustically soft, i.e.,

$$u|_{S_m} = 0 \qquad 1 \le m \le M. \tag{7}$$

As a result of the distribution of many small particles in D, one obtains a new material, which we want to be a "smart" material, that is, a material which has some desired properties. Specifically, we want this material to scatter the incident plane wave according to an a priori given desired radiation pattern. Is this possible? If yes, how does one distribute the small particles in order to create such a material?

We study this problem and solve the following two problems, which can be considered as problems of *nanotechnology*.

The first problem is:

Given an arbitrary function $f(\beta) \in L^2(S^2)$, can one distribute small particles in D so that the resulting medium generates the radiation pattern $A(\beta) := A(\beta, \alpha)$, at a fixed $k > 0$ and a fixed $\alpha \in S^2$, such that

$$\|f(\beta) - A(\beta)\|_{L^2(S^2)} \leq \varepsilon, \tag{8}$$

where $\varepsilon > 0$ is an arbitrary small fixed number?

The answer is yes, and we give an algorithm for calculating such a distribution. This distribution is not uniquely defined by the function $f(\beta)$ and the number $\varepsilon > 0$.

The second problem is:

Given a scattering amplitude $f(\alpha', \alpha)$, corresponding to some refraction coefficient $n(x)$ in a bounded domain D, can one distribute small particles in D so that the resulting medium generates the scattering amplitude $A(\alpha', \alpha)$ such that

$$\|f(\alpha', \alpha) - A(\alpha', \alpha)\|_{L^2(S^2 \times S^2)} \leq \varepsilon, \tag{9}$$

where $\varepsilon > 0$ is an arbitrary small fixed number?

The answer is yes, and we give an algorithm for calculating the density of the desired distribution of small particles given $f(\alpha', \alpha)$, $\forall \alpha', \alpha \in S^2$, $k > 0$ being fixed.

To our knowledge the above two problems have not been studied in the literature. Our solution to these problems is based on some new results concerning the properties of the scattering amplitudes, on our earlier results on wave scattering by small bodies of arbitrary shapes (see Ref. 8), and on our solution of the 3D inverse Schrödinger scattering problem with fixed-energy data,[7] Chapter 5,[5,6]

In Section 2 we derive some *new approximation properties* of the scattering amplitudes. Essentially, we prove the existence of a potential $q \in L^2(D)$ such that the corresponding to this q scattering amplitude $A(\beta)$, $\beta = \alpha'$, at an arbitrary fixed $\alpha \in S^2$ and an arbitrary fixed $k > 0$, approximates with any desired accuracy any given function $f(\beta) \in L^2(S^2)$. Moreover, we give formulas for calculating this q, and these formulas work numerically for arbitrary f (see Ref. 11). The potential q is related explicitly to a certain distribution of small particles in D. Consequently, we give formulas for calculating this distribution.

In Section 3 we derive an equation describing the self-consistent field in the medium consisting of the small particles distributed in D. This equation is equivalent to a Schrödinger equation with a potential $q(x)$ supported in the bounded domain D and related in a simple way to the density of the distribution of the small particles.

The author has solved the 3D inverse scattering problem of finding a compactly supported potential $q \in L^\infty(D)$ from the knowledge of noisy fixed-energy scattering amplitude.[5,6,7] This algorithm allows one to calculate $q_\delta(x)$ from the knowledge of noisy data $f_\delta(\alpha', \alpha)$, $\sup_{\alpha',\alpha\in S^2} |f_\delta - f| \le \delta$, such that

$$\sup_{x\in D} |q_\delta(x) - q(x)| \le \eta(\delta) \underset{\delta\to 0}{\longrightarrow} 0, \tag{10}$$

where $q(x)$ is the exact potential, generating the exact scattering amplitude $f(\alpha', \alpha)$ at a fixed $k > 0$.

Applying this algorithm to the exact data $f(\alpha', \alpha)$ or to the noisy data $f_\delta(\alpha', \alpha)$, one obtains a stable (in the sense (10)) approximation of q, and, consequently, of the density of the distribution of small particles, which generates the scattering amplitude arbitrarily close to the a priori given scattering amplitude.

The author's solution of the 3D inverse scattering problem with the error estimates is described in Section 4.

2. APPROXIMATION PROPERTIES OF THE SCATTERING AMPLITUDES

If $k > 0$ is fixed, then the scattering problem (1)–(3) is equivalent to the Schrödinger scattering problem on the potential $q_0(x)$:

$$[\nabla^2 + k^2 - q_0(x)]u = 0 \quad \text{in } \mathbb{R}^3, \tag{11}$$

$$q_0(x) = \begin{cases} 0 \text{ in } D', & D' := \mathbb{R}^3 \setminus D \\ k^2[1 - n_0(x)] & \text{in } D. \end{cases} \tag{12}$$

The scattering solution $u = u_{q_0}$ solves (uniquely) the equation

$$u_{q_0} = u_0 - \int_D g(x, y)q_0(y)u_{q_0}(y)dy, \quad g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x - y|}. \tag{13}$$

The corresponding scattering amplitude is:

$$A_0(\alpha', \alpha) = -\frac{1}{4\pi}\int_D e^{-ik\alpha'\cdot x} q_0(x)u_{q_0}(x, \alpha)dx, \tag{14}$$

where the dependence on k is dropped since $k > 0$ is fixed.

If q_0 is known, then $A_0 := A_{q_0}$ is known. Let $q \in L^2(D)$ be a potential and $A_q(\alpha', \alpha)$ be the corresponding scattering amplitude. Fix $\alpha \in S^2$ and denote

$$A(\beta) := A_q(\alpha', \alpha), \qquad \alpha' = \beta. \tag{15}$$

Then

$$A(\beta) = -\frac{1}{4\pi}\int_D e^{-ik\beta\cdot x} h(x)dx, \qquad h(x) := q(x)u_q(x, \alpha). \tag{16}$$

Theorem 1. *Let $f(\beta) \in L^2(S^2)$ be arbitrary. Then*

$$\inf_{h \in L^2(D)} \left\| f(\beta) - \left(-\frac{1}{4\pi} \int_D e^{-ik\beta\cdot x} h(x)dx \right) \right\| = 0. \qquad (17)$$

Proof of Theorem 1: If (17) fails, then there is a function $f(\beta) \in L^2(S^2)$, $f \neq 0$, such that

$$\int_{S^2} d\beta f(\beta) \int_D e^{-ik\beta\cdot x} h(x)dx = 0 \qquad \forall h \in L^2(D). \qquad (18)$$

This implies

$$\varphi(x) := \int_{S^2} d\beta f(\beta) e^{-ik\beta\cdot x} = 0 \qquad \forall x \in D. \qquad (19)$$

The function $\varphi(x)$ is an entire function of x. Therefore (19) implies

$$\varphi(x) = 0 \qquad \forall x \in \mathbb{R}^3. \qquad (20)$$

This and the injectivity of the Fourier transform imply $f(\beta) = 0$. Note that $\varphi(x)$ is the Fourier transform of the distribution $f(\beta)\delta(k-\lambda)\lambda^{-2}$, where $\delta(k-\lambda)$ is the delta-function and $\lambda\beta$ is the Fourier transform variable. The injectivity of the Fourier transform implies $f(\beta)\delta(k-\lambda) = 0$, so $f(\beta) = 0$.

Theorem 1 is proved. □

Let us give an algorithm for calculating $h(x)$ in (17) such that the left-hand side of (17) does not exceed ε, where $\varepsilon > 0$ is an arbitrary small given number.

Let $\{Y_\ell(\beta)\}_{\ell=0}^{\infty}$, $Y_\ell = Y_{\ell,m}$, $-\ell \leq m \leq \ell$, be the orthonormal in $L^2(S^2)$ spherical harmonics,

$$Y_{\ell,m}(-\beta) = (-1)^\ell Y_{\ell,m}(\beta), \quad \overline{Y_{\ell,m}(\beta)} = (-1)^{\ell+m} Y_{\ell,-m}(\beta), \qquad (21)$$

$$j_\ell(r) := \left(\frac{\pi}{2r}\right)^{1/2} J_{\ell+\frac{1}{2}}(r), \qquad (22)$$

where J_ℓ are the Bessel functions. It is known that

$$e^{-ik\beta\cdot x} = \sum_{\ell=0, -\ell\leq m\leq \ell} 4\pi(-i)^\ell j_\ell(kr)\overline{Y_{\ell,m}(x^0)} Y_{\ell,m}(\beta), \quad x^0 := \frac{x}{|x|}. \qquad (23)$$

Let us expand f into the Fourier series with respect to spherical harmonics:

$$f(\beta) = \sum_{\ell=0, -\ell\leq m\leq \ell} f_{\ell,m} Y_{\ell,m}(\beta). \qquad (24)$$

Choose L such that

$$\sum_{\ell > L} |f_{\ell,m}|^2 \leq \varepsilon^2. \qquad (25)$$

With so fixed L, take $h_{\ell,m}(r)$, $0 \le \ell \le L$, $-\ell \le m \le \ell$, such that

$$f_{\ell,m} = -(-i)^{\ell}\left(\frac{\pi}{2k}\right)^{1/2}\int_0^b r^{3/2} J_{\ell+\frac{1}{2}}(kr) h_{\ell,m}(r)dr, \tag{26}$$

where $b > 0$, the origin O is inside D, the ball centered at the origin and of radius b belongs to D, and $h_{\ell,m}(r) = 0$ for $r > b$. There are many choices of $h_{\ell,m}(r)$ which satisfy (26). If (25) and (26) hold, then the norm on the left-hand side of (17) is $\le \varepsilon$.

A possible analytical choice of $h_{\ell,m}(r)$ is

$$h_{\ell,m} = \begin{cases} \dfrac{f_{\ell,m}}{-(-i)^{\ell}\sqrt{\frac{\pi}{2k}} g_{1,\ell+\frac{1}{2}}(k)}, & \ell \le L, \\ 0, & \ell > L, \end{cases} \tag{27}$$

where we have assumed that $b = 1$ in (26), and used the following formula (see Ref. 1, formula 8.5.8):

$$\int_0^1 x^{\mu+\frac{1}{2}} J_\nu(kx)dx = k^{-\mu-\frac{3}{2}}\left[\left(\nu+\mu-\frac{1}{2}\right)kJ_\nu(r)S_{\mu-\frac{1}{2},\nu-1}(k)\right.$$
$$\left. - kJ_{\nu-1}(k)S_{\mu+\frac{1}{2},\nu}(k) + 2^{\mu+\frac{1}{2}}\frac{\Gamma\left(\frac{\mu+\nu}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+\frac{1}{4}\right)}\right] := g_{\mu,\nu}(k), \tag{28}$$

where $S_{\mu,\nu}(k)$ are Lommel's functions, $\Gamma(x)$ is the Gamma-function, $h_{\ell,m}(r)$ in (27) do not depend on r, and we assume that $h(x) = 0$ for $r := |x| > 1$.

Let us prove that for any $q \in L^2$ there exists a $q \in L^2(D)$ such that $q(x)u_q$ approximates $h(x)$ in $L^2(D)$-norm with arbitrary accuracy.

Theorem 2. *Let $h \in L^2(D)$ be arbitrary. Then*

$$\inf_{q\in L^2(D)} \|h - qu_q(x,\alpha)\| = 0. \tag{29}$$

Here $\alpha \in S^2$ and $k > 0$ are arbitrary, fixed. There exists a potential $q \in L^2(D)$ such that $h = qu$ provided that the norm $\|h\|_{L^2(D)}$ is sufficiently small.

Proof of Theorem 2: In this proof we first assume that the norm of f is small, and then we drop the "smallness" assumption. If the norm of f is sufficiently small, the norm of h is small, so that the condition

$$\inf_{x\in D}\left|u_0(x) - \int_D g(x,y)h(y)dy\right| > 0 \tag{30}$$

is satisfied. Here g is defined in formula (13). If this condition is satisfied, then the formula

$$q(x) = h(x)\left[u_0(x) - \int_D g(x,y)h(y)dy\right]^{-1} \tag{31}$$

yields the desired potential q. The function h generates the function $u := \frac{h}{q}$, where u is the scattering solution, corresponding to the potential q, constructed by formula (31). Therefore, the infimum in (29) is attained if condition (30) is satisfied by the given h. A sufficient smallness condition for the inequality (30) to hold, is $\int_D \frac{|h(y)|dy}{4\pi|x-y|} < 1$.

If f is arbitrary, not necessarily small, then h is not necessarily small. If, nevertheless, condition (30) holds for this h, then the potential q, given by formula (31), belongs to $L^2(D)$ and yields the scattering amplitude $A_q(\beta)$ which satisfies (8).

On the other hand, if condition (30) does not hold, then formula (31) may yield a potential which is not locally integrable. In this case one can perturb h slightly, so that the perturbed h, denoted by h_δ, $\|h - h_\delta\|_{L^2(D)} < \delta$, would yield, by formula (31) with h_δ in place of h, a potential $q_\delta \in L^2(D)$. If δ is sufficiently small, then this potential generates the scattering amplitude $A_{q_\delta}(\beta)$, which satisfies estimate (8), possibly with $c\varepsilon$ in place of ε, where the positive constant c does not depend on ε.

A proof of the possibility to perturb h so that the the perturbed function h_δ would yield by formula (31) a bounded potential q_δ is given below, in Lemma 1. Theorem 2 is proved. □

Lemma 1. *Assume that h is analytic in D and bounded in the closure of D. Then there exists a small perturbation h_δ of h, $\|h - h_\delta\|_{L^2(D)} < \delta$, such that the function $q_\delta := \frac{h_\delta(x)}{u_0(x) - \int_D g(x,y)h_\delta(y)dy}$ is bounded.*

Proof of Lemma 1: First we note, that one may assume without loss of generality that h is analytic in the closure of D, because analytic functions are dense in $L^2(D)$. One may even assume that h is a polynomial, since polynomials are also dense in $L^2(D)$ if D is bounded. Let

$$\psi := u_0(x) - \int_D g(x,y)h(y)dy.$$

The function ψ is analytic in D since h is. Define the set of its zeros in D:

$$N := \{x : \psi(x) = 0,\ \ x \in D\},$$

and let

$$N_\delta := \{x : |\psi(x)| < \delta,\ \ x \in D\}.$$

Generically, the set N is a line, because it is the intersection of two surfaces: $\psi_1 = 0$ and $\psi_2 = 0$, where $\psi_1 := \operatorname{Re}\psi$ and $\psi_2 := \operatorname{Im}\psi$. Let $D_\delta := D \setminus N_\delta$. Generically, $|\nabla\psi| \geq c > 0$ on N, and, therefore, by continuity, in N_δ. A small perturbation of h will lead to these generic assumptions.

Consider the new coordinates

$$s_1 = \psi_1, \quad s_2 = \psi_2, \quad s_3 = x_3.$$

Choose the origin in N. The Jacobian

$$J := \frac{\partial(s_1, s_2, s_3)}{\partial(x_1, x_2, x_3)}$$

is non-singular in N_δ because $\nabla\psi_1$ and $\nabla\psi_2$ are linearly independent in N_δ. Also we have $\max_{x \in N_\delta}(|J| + |J^{-1}|) < c$. By $c > 0$ various constants independent of δ are denoted. Define $h_\delta = h$ in D_δ and $h_\delta = 0$ in N_δ. Let

$$q_\delta := \frac{h_\delta}{\psi_\delta} \ \text{ in } \ D_\delta, \quad q_\delta := 0 \ \text{ in } \ N_\delta,$$

where

$$\psi_\delta := u_0(x) - \int_D g(x, y) h_\delta(y) dy.$$

Let us prove that the function q_δ is bounded. It is sufficient to check that

$$|\psi_\delta| > c\delta > 0 \quad \text{in} \quad D_\delta.$$

By c we denote various positive constants independent of δ. One has

$$|\psi_\delta| \geq |\psi| - I(\delta) \geq \delta - I(\delta),$$

where

$$I(\delta) := \frac{M}{4\pi} \int_{N_\delta} \frac{dy}{|x - y|}, \quad x \in D_\delta, \quad M = \max_{x \in N_\delta} |\psi|.$$

The constant M does not depend on δ because ψ is bounded in D.

The proof will be completed if we establish the estimate

$$I(\delta) = O(\delta^2 |\ln(\delta)|).$$

Let us derive this estimate. It is sufficient to check this estimate for the integral

$$I := \int_{N_\delta} \frac{dy}{|y|} \leq 2\pi c \int_{\rho \leq \delta} d\rho\rho \int_0^1 \frac{ds_3}{\sqrt{s_3^2 + \rho^2}},$$

where $\rho^2 = s_1^2 + s_2^2$, we have changed the variables y to s, used the estimate $|J^{-1}| < c$, and took into account that the region N_δ is described by the inequalities

$\rho \le \delta$, $0 \le s_3 \le 1$. A direct calculation of the integral I yields the desired estimate: $I = O(\delta^2 |\ln(\delta)|)$.

Lemma 1 is proved. □

Let us give a different, more intuitive, point of view on the role of the "smallness" of h (or of f) assumption. If $\|q\|_{L^2(D)} \to 0$, then the set of functions qu becomes a linear set. Thus, if (29) fails, then there exists an $h \neq 0$, $h \in L^2(D)$ such that

$$\int_D h(x)q(x)u_q(x,\alpha)dx = 0 \qquad \forall q \in L^2(D) \quad \|q\| \ll 1. \tag{32}$$

Condition (32) holds in the limit $\|q\| \to 0$ because in this limit the set of functions qu becomes linear, as one can see from the following argument.

Let $c = const > 0$ be small. We will take $c \to 0$ eventually. Choose

$$q = c\overline{h}e^{-ik\alpha\cdot x}.$$

For sufficiently small $c > 0$ the equation

$$u_q = e^{ik\alpha\cdot x} - \int_D g(x,y)c\overline{h}e^{-ik\alpha\cdot y}u_q dy := e^{ik\alpha\cdot x} - Tu_q$$

is uniquely solvable for u_q in $C(D)$ because $\|T\| < 1$ if $c > 0$ is sufficiently small. We have

$$qu_q = qe^{ik\alpha\cdot x} - qTu_q = c\overline{h} + O(c^2), \quad c \to 0. \tag{33}$$

The above formula explains the meaning of "linearity in the limit of small potentials": the term $O(c^2)$ is negligible when $c \to 0$.

Substitute (33) into (32), divide by c, and take $c \to 0$. The result is:

$$\int_D |h|^2 dx = 0. \tag{34}$$

This implies $h = 0$.

We describe the relation between $q(x)$ and the density distribution of small particles in Section 3. This relation makes it clear that a suitable distribution of small particles will produce any desirable potential $q \in L^2(D)$, and, consequently, any desirable scattering amplitude (radiation pattern) at an arbitrary fixed $\alpha \in S^2$ and $k > 0$.

We describe the algorithm for calculating the above distribution of small particles, given $f(\beta) \in L^2(S^2)$, in Section 3.

3. SCATTERING BY MANY SMALL PARTICLES

The exact statement of the problem is:

$$[\nabla^2 + k^2 - q_0(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \tag{35}$$

$$u = 0 \text{ on } \bigcup_{m=1}^{M} S_m, \quad S_m := \partial D_m. \tag{36}$$

$$u = e^{ik\alpha\cdot x} + v := u_0 + v, \tag{37}$$

$$v = A(\alpha', \alpha)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |\alpha| \to \infty, \quad \alpha' = \frac{x}{r}. \tag{38}$$

We look for the solution of the form

$$u(x) = U_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x, s)\sigma_m(s)ds, \tag{39}$$

where $G(x, s)$ is the Green function which solves the scattering problem in the absence of small particles, i.e.:

$$[\nabla^2 + k^2 - q_0(x)]G(x, y) = -\delta(x - y) \text{ in } \mathbb{R}^3, \tag{40}$$

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial G}{\partial |x|} - ikG\right) = 0, \tag{41}$$

and U_0 is the corresponding scattering solution in the absence of small bodies. It was proved in Ref. [3, p. 46], (see also Ref. 7, p. 264), that

$$G(x, y) = \frac{e^{ik|x|}}{4\pi|x|}U_0(y, \alpha) + o\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \quad \alpha = -\frac{x}{|x|}, \tag{42}$$

where U_0 is the scattering solution corresponding to q_0.

The function (39) solves equation (35) and satisfies the radiation condition (38), because

$$U_0 = u_0 + v_0, \quad u_0 = e^{ik\alpha\cdot x}, \tag{43}$$

where v_0 satisfies the radiation condition (41). Therefore (39) solves the problem (35)–(38) if σ_m are such that the boundary condition (36) is satisfied. All the above arguments did not use the smallness of the particles.

Let us now use the smallness assumptions (4) and (6). Let $x_j \in D_j$ be an arbitrary point inside D_j. Then

$$\sup_{s\in S_j} |G(x,s) - G(x,x_j)| = O\left(ka + \frac{a}{d}\right), \quad |x - x_j| > d. \tag{44}$$

This follows from the integral equation, relating G and g:

$$G(x,y) = g(x,y) - \int_D g(x,z)q_0(z)G(z,y)dz,$$

and from the estimates:

$$\left|\frac{e^{ik|x-s|}}{4\pi|x-s|} - \frac{e^{ik|x-x_j|}}{4\pi|x-x_j|}\right| = \frac{1}{4\pi|x-x_j|}\left|\frac{e^{ik(|x-s|-|x-x_j|)}|x-x_j|}{|x-s|} - 1\right|,$$

$$k\||x-s| - |x-x_j|\| = k|x-x_j|\left(1 + O\left(\frac{a}{d}\right) + O(ka)\right),$$

$$|x-s| = |x-x_j-(s-x_j)| = |x-x_j|\left(1 + O\left(\frac{a}{d}\right)\right).$$

From the integral equation for G it follows that

$$G(x,y) = g(x,y)[1 + O(|x-y|] \quad \text{as} \quad x \to y.$$

Therefore one may approximate (39) as

$$u(x) = U_0(x) + \sum_{m=1}^{M} G(x,x_j)Q_m\left[1 + O\left(ka + \frac{a}{d}\right)\right], \tag{45}$$

where $|x - x_{j_m}| \geq d$ for all m, $1 \leq m \leq M$, and

$$Q_m = \int_{S_m} \sigma_m(s)ds. \tag{46}$$

Therefore, if one knows the numbers Q_m, $1 \leq m \leq M$, then one knows the scattering solution $u(x)$ at any point which is at a distance $\geq d$ from the small body nearest to x.

Generically, $Q_m \neq 0$. However, if the Neumann boundary condition is imposed on S_m, then $Q_m = 0$, and, by this reason, one has to take into account the next non-vanishing term. Such a more delicate analysis is carried over in the problem of wave scattering by a single acoustically hard body in Ref. [8, pp. 98–99].

Let us derive a linear algebraic system for calculating Q_m. To do this, let us use the boundary condition (36). We have:

$$\int_{S_m} G(s,t)\sigma_m dt = -\left[U_0(x_m) + \sum_{j\neq m} G(x_m,x_j)Q_j\right]. \tag{47}$$

Since $k|s-t| \leq 2ka \ll 1$, one has

$$G(s,t) \approx \frac{e^{ik|s-t|}}{4\pi|s-t|} = \frac{1}{4\pi|s-t|}\left(1 + O(ka)\right). \tag{48}$$

Consequently, Eq. (47) can be written as

$$\int_{S_m} \frac{\sigma_m(t)}{4\pi|s-t|} dt = -\left[U_0(x_m) + \sum_{j \neq m} G(x_m, x_j) Q_j\right]. \tag{49}$$

This is an equation for the electrostatic charge distribution σ_m on the surface S_m of the perfect conductor D_m, charged to the potential which is given by the right-hand side of (49). Therefore, the total charge Q_m on the surface S_m of the conductor D_m is given by the formula:

$$\int_{S_m} \sigma_m dt = Q_m = -C_m\left[U_0(x_m) + \sum_{j \neq m} G(x_m, x_j) Q_j\right], \quad 1 \leq m, j \leq M, \tag{50}$$

where C_m is the electrical capacitance of the conductor D_m, and minus the expression in the brackets can be interpreted as the potential to which the conductor D_m is charged. Equation (50) is a linear algebraic system for the unknown quantities Q_j, $1 \leq j \leq M$.

Assume that the distribution of small bodies D_m in D is such that

$$\lim_{M\to\infty} \sum_{D_m \subset \widetilde{D}} C_m = \int_{\widetilde{D}} C(x) dx, \tag{51}$$

where $\widetilde{D}$ is an arbitrary subdomain of D. This means that $C(x)$ is the limiting density of the capacitance per unit volume around an arbitrary point $x \in D$. In other words, one can say that $C(x)dx = \sum_{D_m \subset dx} C_m$, where dx is the element of the volume around the point x. Now, the relation (45) in the limit

$$M \to \infty, \quad ka \to 0, \quad \frac{a}{d} \to 0,$$

takes the form

$$u(x) = U_0(x) - \int_D G(x,y) C(y) u(y) dy, \tag{52}$$

where $C(x)$ is defined in (51).

Note that the relative volume of the small particles, injected into D, is negligible as $\frac{a}{d} \to 0$. Indeed, the number of small particles per unit volume is of the order $O(\frac{1}{d^3})$. The volume of one small particle is of the order $O(a^3)$. Thus, the relative volume of the small particles, that is, the total volume of small particles per unit volume of the material in D, is $O(\frac{a^3}{d^3})$. This quantity tends to zero as

$\frac{a}{d} \to 0$. On the other hand, the electrical capacitance of one conductor with the shape of a small particle is of the order $O(a)$. Therefore, the total electrical capacitance per unit volume is $O(\frac{a}{d^3})$. This quantity has locally, around a point $x \in D$, a finite non-zero limit $C(x)$ as $\frac{a}{d} \to 0$, according to (51). Thus, the injection of small particles under our assumptions is similar to "dusting", since the relative volume of the injected particles is negligible. This conclusion should be of interest to experimentalists who will implement practically the theory, developed in this paper.

An equation which is similar to (52), with $g(x_j, y)$ in place of $G(x_j, y)$, has been derived in Ref. 2 by a different argument and in Ref. [3, pp. 191–192], by an argument, close to the one used above (see also Ref. 12).

Equation (52) is equivalent to the Schrödinger equation

$$\left[\nabla^2 + k^2 - q_0(x) - C(x)\right]u = 0, \tag{53}$$

and $u(x)$ is the scattering solution corresponding to the potential

$$q(x) = q_0(x) + C(x). \tag{54}$$

To verify this, one applies the operator $\nabla^2 + k^2 - q_0(x)$ to both sides of Eq. (52) and takes into account Eq. (40). If $q_0(x)$ is known (which we assume), then $q(x)$ and $C(x)$ are in one-to-one correspondence.

If the small particles D_m are identical, and C_0 is the electrical capacitance of a single particle, then

$$C(x) = N(x)C_0, \tag{55}$$

where $N(x)$ is the density of the number of particles in a neighborhood of the point x, that is, the number of particles per unit volume around point x.

Therefore, given $f(\beta) \in L^2(S)$, one found $q(x)$, such that $\|A_q(\beta) - f(\beta)\| \leq \varepsilon$, where $A_q(\beta)$ is the scattering amplitude, corresponding to the potential q, the energy $k^2 > 0$ and the incident direction α being fixed, and $\beta = \alpha'$ is the direction of the scattered wave.

Let us describe the steps of our algorithm.

Step 1. *Given $f(\beta)$, find $h \in L^2(D)$.*

This problem is ill-posed. It is similar to solving first kind integral equation

$$f(\beta) = -\frac{1}{4\pi}\int_D e^{-ik\beta\cdot x}h(x)dx.$$

If this equation is solvable, then f has to be infinitely smooth on S^2. If f_δ is a slightly perturbed f, then the above equation may be not solvable. In Step 1 one solves a problem of approximation of an arbitrary $f \in L^2(S^2)$ by an infinitely smooth function. If f is not continuous, then the increase of the accuracy of approximation results in the growth of the norm $\|h\|_{L^2(D)}$. This would lead to large maximal values of the corresponding q. Therefore a regularization procedure is needed in numerical implementation of our solution. Some related numerical experiments are described in Ref. 11.

Step 2. *Given $h \in L^2(D)$, find q such that $\|h - q(x)u_q(x)\|_{L^2(D)} \leq \varepsilon$.*

Let us elaborate on Step 2. First, assume the existence of a potential q, such that $h = qu$. Consider the equation

$$u = u_0 - \int_D gqu_q dy = u_0 - \int_D ghdy. \tag{56}$$

We have

$$qu_q := h.$$

Thus,

$$A_q(\beta) = -\frac{1}{4\pi}\int_D e^{-ik\beta\cdot x}h(x)dx. \tag{57}$$

Multiply (56) by q. Then

$$h = u_0 q - q\int_D ghdy.$$

Therefore, if

$$\inf_{x\in D}\left|u_0(x) - \int_D g(x,s)h(y)dy\right| > 0,$$

then the solution of the equation $qu_q = h$ is unique and is given by the formula:

$$q(x) = \frac{h(x)}{u_0(x) - \int_D ghdy}. \tag{58}$$

Formula (58) yields a potential for which $A_q(\beta)$ is given by formula (57), and the corresponding scattering solution is given by formula (56). All this is true provided, for example, that

$$\sup_{x\in D}\left|\int_D g(x,y)h(y)dy\right| < 1. \tag{59}$$

Inequality (59) holds if h is fixed and diam D is sufficiently small, because of the following estimate:

$$\sup_x\left|\int_D g(x,y)h(y)dy\right| \leq (4\pi)^{-\frac{1}{2}}\|h\|_{L^2(D)}(\text{diam } D)^{\frac{1}{2}}.$$

Inequality (59) also holds if $\|h\|_{L^2(D)}$ is sufficiently small and D is fixed. The norm $\|h\|_{L^2(D)}$ is small if $\|f\|_{L^2(S^2)}$ is sufficiently small. For the formula (58) to yield the desired potential, the inequality (59) is not necessary. If one can find a potential $q(x)$ from the given h by formula (58), then this q generates the scattering solution by the formula

$$u_q = u_0 - \int_D ghdy, \tag{60}$$

and

$$h = q(x)u_q(x). \tag{61}$$

The potential q can be found by formula (58), provided that $f(\beta)$ is sufficiently small, because then h will be sufficiently small as follows, e.g., from (27).

If q is found, then

$$N(x) = \frac{q(x) - q_0(x)}{C_0}, \tag{62}$$

where C_0 is the electrical capacitance of a conductor with the shape of a small particle, and all small particles are assumed to have the same shape and, therefore, the same electrical capacitance. Thus, the corresponding distribution density of small particles is given analytically.

Analytical formulas, which allow one to calculate C_0 with any desired accuracy, are derived in Ref. 8, see also formula (91) below.

Remark 31. If $f(\beta)$ corresponds to a real-valued $q(x)$, then formula (58) yields a real-valued potential. In general, formula (58) yields a complex-valued potential. To get a complex-valued potential by a formula, similar to (55), one has to replace the Dirichlet boundary condition (36) by the impedance boundary condition

$$u_N = \zeta u \quad on \quad S_m, \tag{63}$$

where N is the exterior unit normal to the boundary S, and ζ is a complex constant, the impedance. Then C_0 in (55) should be replaced by the quantity:

$$C_\zeta = \frac{C_0}{1 + \frac{C_0}{\zeta S}}, \tag{64}$$

(see Ref. [8, pp. 96–97]), and, therefore, formula (55) yields a complex-valued potential $C_\zeta(x)$ if ζ is a complex number.

Suppose that a given h corresponds to a potential $q(x) \in L^2(D)$ in the sense that $h = q(x)u(x)$, where $u(x)$ is the scattering solution corresponding to this $q(x)$ at the wavenumber $k > 0$ and with the incident direction α. Then formula (58) defines $q(x)$, and the corresponding scattering solution is $u = \frac{h(x)}{q(x)}$.

If formula (58) does not produce a $p \in L^2(D)$, then one can replace h in (58) by an h_ε, $\|h - h_\varepsilon\|_{L^2(D)} < \varepsilon$, and get a square-integrable potential q_ε by formula

(58) with h replaced by h_ε. If ε is sufficiently small, this potential q_ε generates the radiation pattern, which differs by $O(\varepsilon)$ from the desired f.

4. RAMM'S SOLUTION OF THE 3D INVERSE SCATTERING PROBLEM WITH FIXED-ENERGY DATA

We follow Refs. 6 and 7. Consider first the inversion of the exact data $A_q(\alpha', \alpha)$.

Let

$$A_q(\alpha', \alpha) = \sum_{\ell,\ell'=0}^{\infty} A_{\ell,m,\ell',m'} Y_{\ell',m'}(\alpha') Y_{\ell,m}(\alpha). \tag{65}$$

It is proved in Ref. [7, p. 262], that

$$|Y_\ell(\theta)| \le \frac{1}{\sqrt{4\pi}} \frac{e^{r|\operatorname{Im}\theta|}}{|j_\ell(r)|}, \quad \forall r > 0, \quad \theta \in \mathcal{M}, \tag{66}$$

where $j_\ell(r)$ is the spherical Bessel function, $\operatorname{Im}\theta$ is the imaginary part of the complex vector $\theta \in \mathcal{M}$, and the algebraic variety $\mathcal{M}$ is defined by the formula:

$$\mathcal{M} = \{z : z \in \mathbb{C}^3, z \cdot z = k^2\}, \quad z \cdot \zeta := \sum_{j=1}^{3} z_j \zeta_j.$$

Estimate (66) allows one to prove (see Ref. 6) that the series (65) converges absolutely for $\alpha' = \theta' \in \mathcal{M}$, so that the exact data $A_q(\alpha', \alpha)$ allow one to calculate the values $A_q(\theta', \alpha), \theta' \in \mathcal{M}$. These values are used below in the inversion formula (68).

One can prove Ref. [7, p. 258], that any $\xi \in \mathbb{R}^3$ can be written (nonuniquely) as

$$\xi = \theta' - \theta, \quad \theta', \theta \in \mathcal{M}, \quad |\theta| \to \infty. \tag{67}$$

In Ref. [7, p. 258], explicit analytical formulas are given for θ' and θ satisfying (67).

The exact data $A(\alpha', \alpha)$ admit an analytic continuation from $S^2 \times S^2$ onto $\mathcal{M} \times S^2$. Let

$$\widetilde{q}(\xi) := \int_D q(x) e^{-i\xi\cdot x} dx.$$

The inversion formula, proved in Ref. [7, pp. 264–266], is

$$\widetilde{q}(\xi) = \lim_{\substack{|q|\to\infty \\ \theta'-\theta=\xi, \theta',\theta\in\mathcal{M}}} \left[-4\pi \int_{S^2} A(\theta', \alpha) \nu(\alpha, \theta) d\alpha \right], \tag{68}$$

where (67) holds and $\nu(\alpha, \theta)$ is an arbitrary approximate solution to the problem

$$\mathcal{F}(\nu) := \int_{a_1 \le |x| \le b} |\rho(x)|^2 dx = \inf := d(\theta). \tag{69}$$

Here

$$\rho(x) := e^{-i\theta \cdot x} \int_{S^2} u(x, \alpha)\nu(\alpha, \theta) d\alpha - 1, \tag{70}$$

$a_1 > 0$ is a radius of a ball which contains D as a strictly inner-subdomain, and $b > a_1$ is an arbitrary fixed number, and $u(x, \alpha)$ is the scattering solution. The approximate solution ν to (69) is understood in the following sense:

$$\mathcal{F}(\nu) \le 2d(\theta). \tag{71}$$

This means that it is not necessary to find a very accurate approximation of the infimum in the minimization problem (69). It is sufficient, for example, to find any function $\nu(\alpha, \theta)$ for which the functional (69) takes the value not more than $2d(\theta)$. The inversion formula (68) holds with such $\nu(\alpha, \theta)$. The inversion formula (73) below is given with the error term.

It is proved in Ref. [7, p. 266], that

$$d(\theta) \le \frac{c}{|\theta|}, \quad \theta \in \mathcal{M}, \tag{72}$$

where $c = c(\|q\|) > 0$ is a constant depending on an $L^\infty(D)$ norm of q. Therefore, given the exact data $A_q(\alpha', \alpha)$, one recovers the potential $q(x)$ by formula (68).

The error estimate of formula (68) is given by the formula:

$$\widetilde{q}(\xi) = -4\pi \int_{S^2} A(\theta', \alpha)\nu(\alpha, \theta) d\alpha + O\left(\frac{1}{|\theta|}\right), \quad |\theta| \to \infty, \tag{73}$$

where (67) holds.

If $q(x)$ is found, then

$$N(x) = \frac{q(x) - q_0(x)}{C_0}, \tag{74}$$

so that the density of distributions of small particles is found analytically, explicitly.

Consider now the inversion of noisy data $A_\delta(\alpha', \alpha)$,

$$\sup_{\alpha', \alpha \in S^2} |A_\delta(\alpha', \alpha) - A(\alpha', \alpha)| \le \delta. \tag{75}$$

Here the exact data $A(\alpha', \alpha)$ corresponds to an exact potential and is assumed not known. Instead, its noisy measurements $A_\delta(\alpha', \alpha)$ are assumed known.

Define

$$N(\delta) = \left[\frac{|\ln \delta|}{\ln|\ln \delta|}\right], \tag{76}$$

where $[x]$ is the integer nearest to $x > 0$, so that $\lim_{\delta \to 0} N(\delta) = \infty$.

Define also the following objects:

$$\widehat{A}_\delta(\theta', \alpha) = \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\theta'), \quad \sum_\ell := \sum_\ell \sum_{-\ell\le m\le \ell}, \tag{77}$$

$$u_\delta(x, \alpha) = e^{ik\alpha\cdot x} + \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\alpha') h_\ell(kr), \quad \alpha' := \frac{x}{r}, \quad r = |x|, \tag{78}$$

where $h_\ell(kr)$ are the spherical Hankel functions,

$$\rho_\delta(x; \nu) = e^{-i\theta\cdot x} \int_{S^2} u_\delta(x, \alpha)\nu(\alpha) d\alpha - 1, \quad \theta \in \mathcal{M}, \tag{79}$$

$$\mu(\delta) = e^{-\gamma N(\delta)}, \qquad \gamma = \ln \frac{a_1}{b_0} > 0, \tag{80}$$

$$b_0 := \frac{1}{2} \operatorname{diam} D, \qquad \kappa = |\operatorname{Im} \theta|. \tag{81}$$

Let

$$b_0 < a_1 < b, \tag{82}$$

where a_1 and b are arbitrary positive fixed numbers. Consider the problem:

$$|\theta| = \sup := \vartheta(\delta) \tag{83}$$

under the constraints

$$|\theta| \left[\|\rho_\delta(\nu)\|_{L^2(\{x: a_1 \le |x| \le b\})} + \|\nu\|_{L^2(S^2)} e^{\kappa b} \mu(\delta) \right] \le c, \tag{84}$$

$$\theta \in \mathcal{M}, \quad \theta' - \theta = \xi, \quad \theta', \theta \in \mathcal{M}, \tag{85}$$

where $c > 0$ is a sufficiently large constant, and $b_0 < a_1 < b$.

It is proved in Ref. [7, p. 271], that

$$\vartheta(\delta) = O\left(\frac{|\ln \delta|}{(\ln|\ln \delta|)^2} \right) \qquad \delta \to 0. \tag{86}$$

Let $\theta(\delta)$ and $\nu_\delta(\alpha)$ be any approximate solution to (83)–(85) in the sense that

$$|\theta(\delta)| \ge \frac{1}{2} \vartheta(\delta). \tag{87}$$

Define

$$\widehat{q}_\delta := -4\pi \int_{S^2} A_\delta(\theta', \alpha) \nu_\delta(\alpha) d\alpha. \tag{88}$$

The following result is proved in Ref. [7, p. 271].

Theorem (Ramm) *One has*

$$\sup_{\xi\in\mathbb{R}^3} |\widehat{q}_\delta - \widetilde{q}(\xi)| = O\left(\frac{(\ln|\ln\delta|)^2}{|\ln\delta|}\right), \quad \delta\to 0. \tag{89}$$

This result gives an inversion formula for finding the potential from noisy fixed-energy scattering data.

Thus, the algorithm for finding the density of the distribution of small particles from the fixed-energy scattering data $A(\alpha', \alpha)$ can be formulated as follows:

Step 1. *Given $A(\alpha', \alpha)$, find $q(x)$ using the inversion formulas (68) in the case of the exact data or (88) in the case of noisy data.*

Step 2. *Find the density of the distribution of the small particles by formula (62), where formulas for C_0 are given in Ref. [8, p. 26]:*

$$\left|C_0 - C^{(n)}\right| = O(Q^n), \quad 0 < Q < 1, \tag{90}$$

where Q depends only on the geometry of the surface,

$$C^{(n)} = 4\pi|S|^2 \left\{ \frac{(-1)^n}{(2\pi)^n} \int_S \int_S \frac{dsdt}{r_{st}} \underbrace{\int_S \cdots}_{n \text{ integrals}} \right.$$
$$\left. \times \int_S \psi(t, t_1)\ldots\psi(t_{n_1}, t_n) dt_1 \cdot dt_n \right\}^{-1} \tag{91}$$

$$\psi(t,s) = \frac{\partial}{\partial N_t}\frac{1}{r_{st}}, \quad r_{st} = |s-t|, \quad |S| = meas\, S, \tag{92}$$

S is the surface of the conductor, C_0 is the electrical capacitance of this conductor, and N_t is the exterior normal to S at the point t.

In particular, for $n = 0$ one gets

$$C^{(0)} = \frac{4\pi|S|^2}{J}, \qquad J := \int_S \int_S \frac{dsdt}{r_{st}}. \tag{93}$$

It is proved in Ref. [8, p. 30], that

$$C^{(0)} \le C_0. \tag{94}$$

Formula (91) given an approximate value $C^{(n)}$ of the electrical capacitance of a perfect conductor placed in the space with dielectric permittivity $\varepsilon_0 = 1$. If $\varepsilon_0 \neq 1$, then one has to multiply the right-hand side of (91) by ε_0.

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